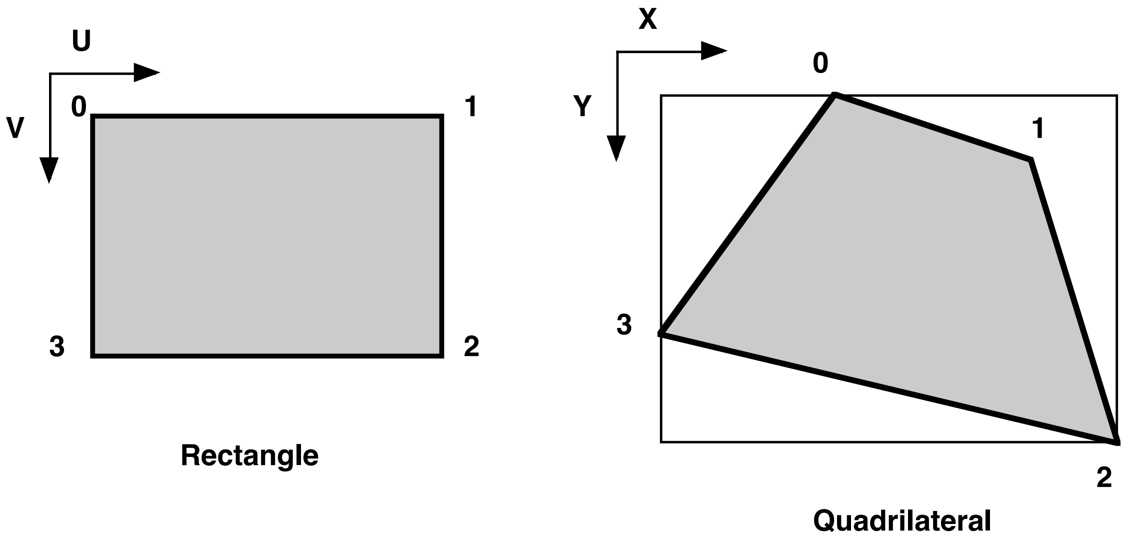
Bilinear Image Warping

Consider the following diagram:



We desire to warp the rectangle into the quadrilateral.

Now the equations that describe transforming the corners of the rectangle to the corresponding corners of the quadrilateral are defined by the following "1.5 order" polynomial:

1a) $X = a_0 + a_1U + a_2V + a_3UV$

1b) $Y = b_0 + b_1U + b_2V + b_3UV$

So for any corner, we substitute the rectangle coordinates U,V into this equation and it gives us the X,Y coordinates of the corresponding corner provided we know the coefficients.

Alternately, one can compute the coefficients given the four corner U,V coordinates in the rectangle (which are found from its dimensions) and the corresponding four X,Y vertices of the quadrilateral.

The corners of the rectangle of dimensions NU x NV are:

Pt 0: (0,0)
Pt 1: (NU-1,0)
Pt 2: (NU-1,NV-1)
Pt 3: (0,NV-1)

Given corresponding coordinates for the quadrilateral as:

Pt 0: (X0,Y0)
Pt 1: (X1,Y1)
Pt 2: (X2,Y2)
Pt 3: (X3,Y3)

If we substitutue Pt 0 into equation 1), we get:

$$a_0 = X_0$$

$$b_0 = Y_0$$

If we substitute Pt 1 into equation 1), we get:

$$a_1 = (X_1 - X_0)/(NU - 1)$$

$$b_1 = (Y_1 - Y_0)/(NU - 1)$$

If we substitute Pt 3 into equation 1), we get:

$$a_2 = (X_3 - X_0)/(NV - 1)$$

$$b_2 = (Y_3 - Y_0)/(NV - 1)$$

If we substitute Pt 2 into equation 1), we get:

$$a_3 = (X_2 - X_1 - X_3 + X_0)/((NU - 1)(NV - 1))$$

$$b_3 = (Y_2 - Y_1 - Y_3 + Y_0)/((NU - 1)(NV - 1))$$

So as long as we know the coefficients, this works fine for projection individual points from the rectangle to the quadrilateral. But when there is scale change, especially magnification, so that the quadrilateral is much larger, then projecting each pixel in the rectangle will leave holes in the quadrilateral.

Thus we want to perform an inverse (or reverse) transformation so that we sequence through every (X,Y) pixel in the quadrilateral and find its corresponding (U,V) coordinate in the rectangle. Then interpolate the neighboring values and put the resultant value at the specified (X,Y) location in the quadrilateral.

To do this we have to find the inverse of equation 1). However, this is not trivial as the polynomial is higher than first degree. It has the cross term in it, but no second order term in either U or V. That is why it is sometimes called a 1.5 order polynomial equation.

So we desire to invert this equation into a form as follows:

2a) $U = \mathrm{F}(X, Y)$

2b) $V = \mathrm{G}(X, Y)$

In order to do this we can solve for either U or V in equation 1a), then substitute that back into equation 1b). It turns out that the former approach is better than the latter as will be explained later.

Lets do the better one, namely, solve for U from equation 1a).

3) $U = (X - a_0 - a_2 V)/(a_1 + a_3 V)$

Substituting this into equation 1b), rationalizing the denominators and combining like powers of V, we find the following quadratic equation that must be solved to get V

4) $AV^2 + BV + C = 0$

where

$$A = (b_2 a_3 - b_3 a_2)$$

$$C = C_1 + (b_1 X - a_1 Y)$$

where

$C_1 = (b_0 a_1 - b_1 a_0)$ is a constant independent of X or Y

$$B = B1 + (b_3 X - a_3 Y)$$

where

$B1 = (b_0 a_3 - b_3 a_0) + (b_2 a_1 - b_1 a_2)$ is a constant independent of X or Y

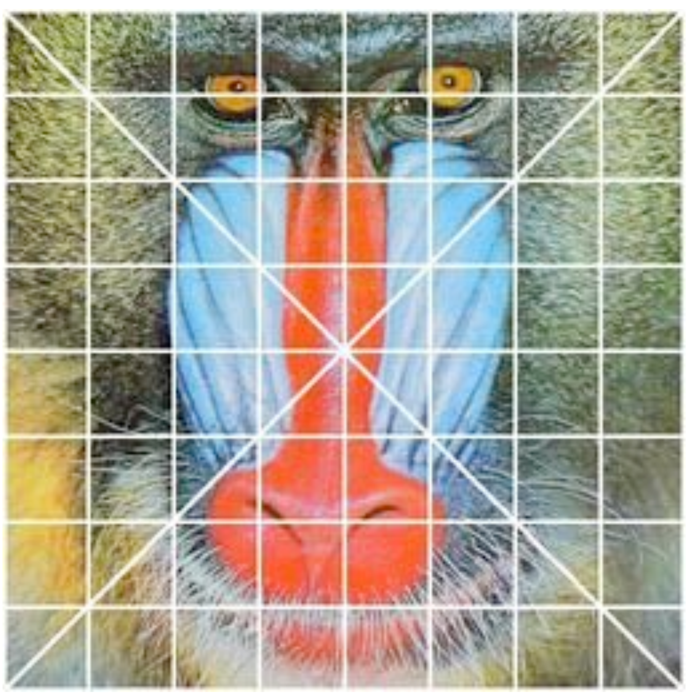
The solution to 4) is of course,

5) $$V = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

This has two solutions, but it turns out that the plus sign is the correct solution here.

Thus equations 4) and 5) properly invert the 1.5 order polynomial and satisfy equation 2)

Here is an example starting with a 256 x 256 image of the mandril which has a grid superimposed.



Using quadrilateral coordinates of:

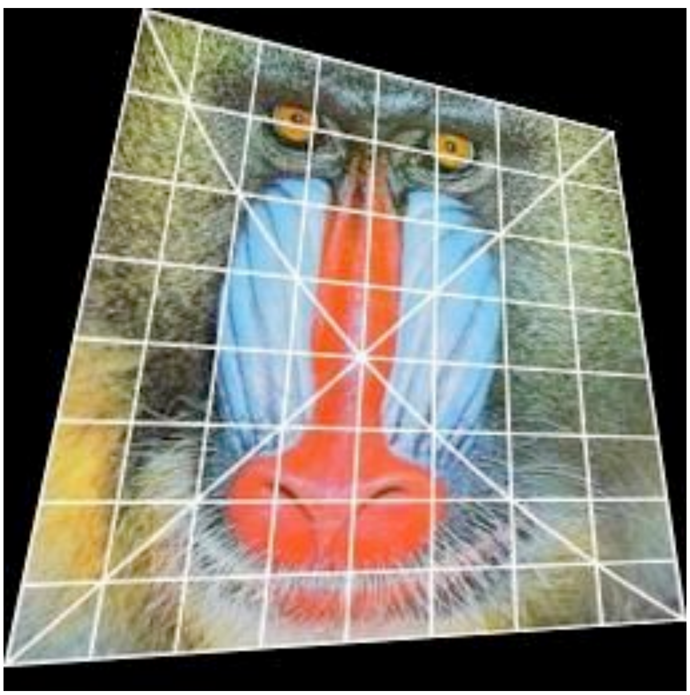
X0=52
Y0=0

X1=228
Y1=46

X2=255
Y2=229

X3=0
Y3=246

The bilinear warp using equations 4) and 5) produces the following image:



Note that a property of the bilinear warp is that lines parallel to the coordinate axes of the input rectangle image remain straight lines in the output quadrilateral image. Thus the quadrilateral boundary edges are straight and the horizontal and vertical grid lines from the rectangle remain straight lines in the quadrilateral. However, one or both diagonals may end up curved depending upon the transformation. They are not necessarily preserved as straight line.

Now we could have chosen to solve for V in equation 1a), resulting in

6) $V = (X - a_0 - a_1 U)/(a_2 + a_3 U)$

which then leads to a quadratic equation of the form

7) $AU^2 + BU + C = 0$

where

$A = (b_1 a_3 - b_3 a_1)$

$C = C_1 + (b_2 X - a_2 Y)$

where

$B = B1 + (b_3 X - a_3 Y)$

where

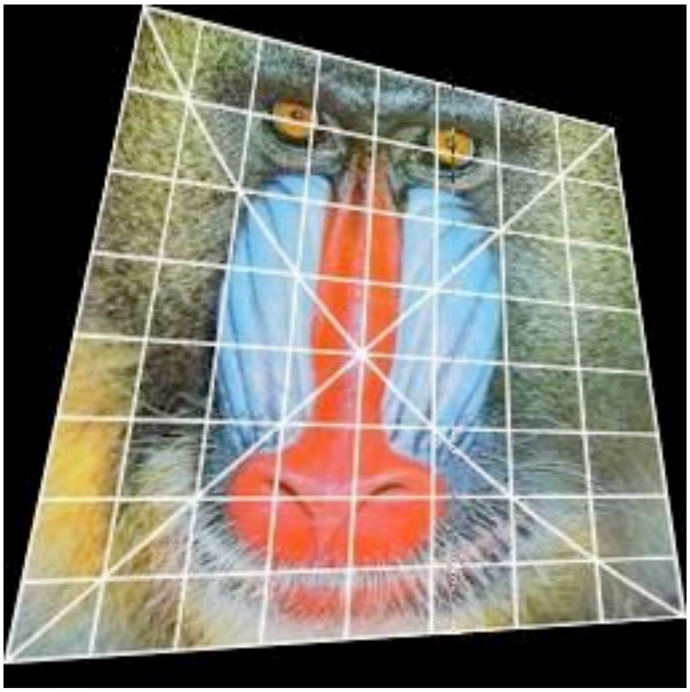
$B1 = (b_0 a_3 - b_3 a_0) + (b_1 a_2 - b_2 a_1)$ is a constant independent of X or Y

And the solution of equation 7) is

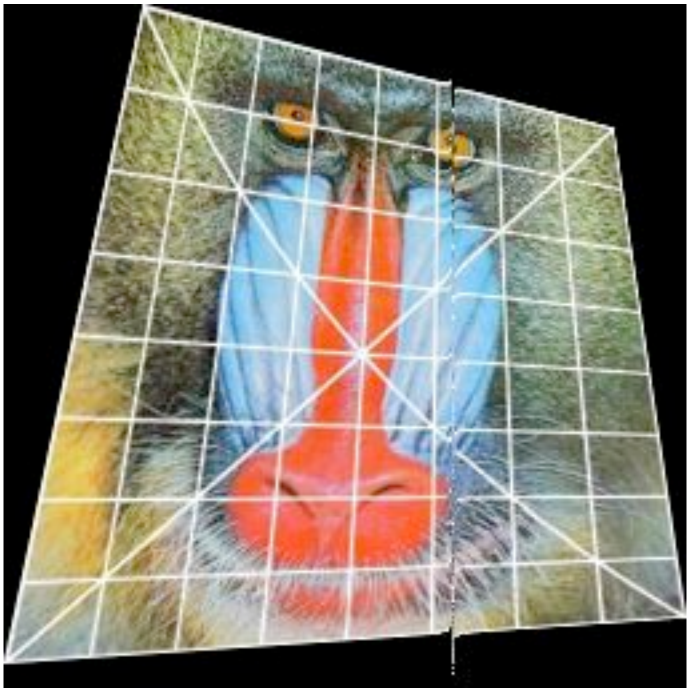
8) $U = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

where the minus sign is the correct choice.

However, the quality of the result for the same quadrilateral vertices is not as good as for the previous result:



Looking carefully, one can see a vertical line through the center of the eye on the right side of the image (which actually extends down the image but is less noticeable). This can be accentuated by reducing the precision of the set of (a,b) coefficients as is shown here:



This seems to occur from a degeneracy in equation 6). By examining the coefficients of equation 6), when X=U, V results a constant independent of U. It is not clear how universal this issue is. Further study is warranted.